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PROGRESS REPORT.

The following gentlemen have been elected members of the Society :

1. Mr. Narbheram P. Pandya, B.A., (Bombay), L.T., (Madras) —Teacher, Baroda High School, Baroda.
2. Mr. S. Venkobachariar, B.A., F.M.U., Principal, The Madura College, Madura.
3. Mr. C. Ganpati Aiyar, B.A , L.T., Mathematics Lecturer, The Madura College, Madura.
4. Mr. S. Mahadeva Aiyar, B.A., L.T., Lecturer in Mathematics, American College, Madura.
5. Mr. K. Srinivasiengar, B.A., L.T., Mathematics Assistant, Setupati High School, Madura, (at concessional rate).
6. Mr. M. Subramaniam, B.A., B.L., High Court Vakil, 28 Kollavari Agraharam, Madras, N.E.

2. The Committee cannot but mention with regret the sad death of H. H. The Maharaja Sri Ramachandra Bharjdeva, of Mourbhanj Baripada (Bengal), who honoured the society by enrolling himself, first as a member, and subsequently as a life member.

3. Through oversight it was omitted to mention in the last Progress Report that Messrs. K. Balaram Iyer, B.A. and R. Swaminatha Iyer, B.A. have taken their seats on the Committee in place of Diwan Bahadur R. Ramachandra Rao, B.A. and Mr. T. R. Venkataswami Naidu, B.A. Prof. R. N. Apte, M.A., L.L.B., F.R.A.S. has also taken over the charge of the Office of Presidentship, from Prof. B. Hanumantha Rau, B.A. The best thanks of the Committee are due to these retiring gentlemen who have worked zealously on the Committee ever since the foundation of the Society.

4. According to article X of our constitution the audited Balance Sheet for the year 1911, is annexed for the information of the General Body. From it, it will be seen that the Society is actually in debt, at the end of the year. This unsatisfactory state of the finances of the Society is due to the fact of the subscriptions not having been received from the members as they should have been. The Committee have consequently gone carefully through this question of finances, and finally resolved upon the adoption of the following Bye-law, to avoid as far as possible a repetition of a similar state of affairs in the future :—

> Bye-Law IX—When the subscription for the year has not been paid by any ordinary member by the end of July in every year, the next number of the Journal shall be sent to him by V.P.P. for the amount of the subscription ; and in the case of a new member, the first number of the Journal shall be sent to him by V.P.P. for the amount of his subscription for the year.

5. The Budget for the current year is also published in the annexed sheet for information.

POONA,
1st April 1912.

D. D. KAPADIA,
Hony. Joint Secretary.

The Theory of Moving Axes.

By A. C. L. Wilkinson, M.A., F.R.A.S.

(*Continued from page* 184, *Vol. III.*)

§ 34. Consider the families of surfaces

$$f_1(x,y,z)=\lambda,\ f_2(x,y,z)=\mu,\ f_3(x,y,z)=\nu,$$

or

$$x=\phi(\lambda,\mu,\nu),\ y=\psi(\lambda,\mu,\nu)\ \ z=\chi(\lambda,\mu,\nu)$$

If μ, ν are constant, we have a curve the direction cosines of whose tangent are proportional to $\left(\frac{\partial x}{\partial\lambda},\frac{\partial y}{\partial\lambda},\frac{\partial z}{\partial\lambda}\right)$.

Similarly, if (λ, ν) and (λ, μ) be constant, we have curves whose tangents have direction cosines proportional to

$$\left(\frac{\partial x}{\partial\mu},\frac{\partial y}{\partial\mu},\frac{\partial z}{\partial\mu}\right) \text{ and } \left(\frac{\partial x}{\partial\nu},\frac{\partial y}{\partial\nu},\frac{\partial z}{\partial\nu}\right).$$

Suppose these three directions are mutually orthogonal, so that

$$\sum\frac{\partial x}{\partial\mu}\frac{\partial x}{\partial\nu}=0,\ \sum\frac{\partial x}{\partial\lambda}\frac{\partial x}{\partial\nu}=0,\ \sum\frac{\partial x}{\partial\lambda}\frac{\partial x}{\partial\mu}=0.$$

Differentiating these with respect to λ, μ, ν we have

$$\sum\frac{\partial^2 x}{\partial\lambda\partial\mu}\frac{\partial x}{\partial\nu}+\sum\frac{\partial x}{\partial\mu}\frac{\partial^2 x}{\partial\lambda\partial\nu}=0$$

$$\sum\frac{\partial^2 x}{\partial\lambda\partial\mu}\frac{\partial x}{\partial\nu}+\sum\frac{\partial x}{\partial\lambda}\frac{\partial^2 x}{\partial\mu\partial\nu}=0$$

$$\sum\frac{\partial^2 x}{\partial\lambda\partial\nu}\frac{\partial x}{\partial\mu}+\sum\frac{\partial x}{\partial\lambda}\frac{\partial^2 x}{\partial\mu\partial\nu}=0$$

whence

$$\sum\frac{\partial x}{\partial\lambda}\frac{\partial^2 x}{\partial\mu\partial\nu}=\sum\frac{\partial x}{\partial\mu}\frac{\partial^2 x}{\partial\lambda\partial\nu}=\sum\frac{\partial x}{\partial\nu}\frac{\partial^2 x}{\partial\lambda\partial\mu}=0.$$

We can now obtain Dupin's theorem that when three surfaces cut orthogonally the curves of intersection are lines of curvature.

We have from the condition the curves of intersection are orthogonal

$$\frac{\frac{\partial x}{\partial\nu}}{\frac{\partial y}{\partial\lambda}\frac{\partial z}{\partial\mu}-\frac{\partial z}{\partial\lambda}\frac{\partial y}{\partial\mu}}=\frac{\frac{\partial y}{\partial\nu}}{\frac{\partial z}{\partial\lambda}\frac{\partial x}{\partial\mu}-\frac{\partial z}{\partial\mu}\frac{\partial x}{\partial\lambda}}=\frac{\frac{\partial z}{\partial\nu}}{\frac{\partial x}{\partial\lambda}\frac{\partial y}{\partial\mu}-\frac{\partial y}{\partial\lambda}\frac{\partial x}{\partial\mu}}$$

Consider the surface $\nu=$constant, or

$$x=\phi(\lambda,\mu),\ y=\psi(\lambda,\mu),\ z=\chi(\lambda,\mu)$$

The tangent plane is

$$A(X-x)+B(Y-y)+C(Z-z)=0,$$

where $$\sum A\frac{\partial x}{\partial \lambda}=0, \sum A\frac{\partial x}{\partial \mu}=0,$$

whence $$A: : \frac{\partial y}{\partial \lambda}\frac{\partial z}{\partial \mu}-\frac{\partial z}{\partial \lambda}\frac{\partial y}{\partial \mu} : : \frac{\partial x}{\partial \nu}.$$

Thus the direction cosines of the normal are proportional to

$$\frac{\partial x}{\partial \nu}, \frac{\partial y}{\partial \nu}, \frac{\partial z}{\partial \nu}.$$

Now, if $x=\phi(\lambda)$, $y=\psi(\lambda)$, $z=\chi(\lambda)$, where μ, ν are constant, is a line of curvature, we must have

$$\begin{vmatrix} dx, & U, & dU \\ dy, & V, & dV \\ dz, & W, & dW \end{vmatrix}=0,$$

where U, V, W are proportional to the direction cosines of the normal to the surface $\nu=$constant.

Hence $$\begin{vmatrix} \frac{\partial x}{\partial \lambda}, & \frac{\partial x}{\partial \nu}, & \frac{\partial^2 x}{\partial \nu \partial \lambda} \\ \frac{\partial y}{\partial \lambda}, & \frac{\partial y}{\partial \nu}, & \frac{\partial^2 y}{\partial \nu \partial \lambda} \\ \frac{\partial z}{\partial \lambda}, & \frac{\partial z}{\partial \nu}, & \frac{\partial^2 z}{\partial \nu \partial \lambda} \end{vmatrix}=0,$$

which reduces to $$\sum \frac{\partial^2 x}{\partial \nu \partial \lambda}\frac{\partial x}{\partial \mu}=0,$$

since $$\frac{\partial y}{\partial \lambda}\frac{\partial z}{\partial \nu}-\frac{\partial y}{\partial \nu}\frac{\partial z}{\partial \lambda} : : \frac{\partial x}{\partial \mu}.$$

This proof is given by Picard, Traité D'Analyse, Tome I.

If λ, μ, ν simultaneously vary, we have

$$ds^2=dx^2+dy^2+dz^2=\left\{\left(\frac{\partial x}{\partial \lambda}\right)^2+\left(\frac{\partial y}{\partial \lambda}\right)^2+\left(\frac{\partial z}{\partial \lambda}\right)^2\right\} d\lambda^2+\dots$$

where the coefficients of $d\lambda d\mu$, $d\lambda d\nu$, $d\mu d\nu$ are zero in virtue of the conditions of orthogonality.

MADRAS

We write $$ds^2 = A^2 d\lambda^2 + B^2 d\mu^2 + C^2 d\nu^2$$

where
$$A^2 = \left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 + \left(\frac{\partial z}{\partial \lambda}\right)^2$$
$$B^2 = \left(\frac{\partial x}{\partial \mu}\right)^2 + \left(\frac{\partial y}{\partial \mu}\right)^2 + \left(\frac{\partial z}{\partial \mu}\right)^2$$
$$C^2 = \left(\frac{\partial x}{\partial \nu}\right)^2 + \left(\frac{\partial y}{\partial \nu}\right)^2 + \left(\frac{\partial z}{\partial \nu}\right)^2$$

The three straight lines whose direction cosines are

$$\frac{1}{A}\frac{\partial x}{\partial \lambda},\ \frac{1}{A}\frac{\partial y}{\partial \lambda},\ \frac{1}{A}\frac{\partial z}{\partial \lambda}$$
$$\frac{1}{B}\frac{\partial x}{\partial \mu},\ \frac{1}{B}\frac{\partial y}{\partial \mu},\ \frac{1}{B}\frac{\partial z}{\partial \mu}$$
$$\frac{1}{C}\frac{\partial x}{\partial \nu},\ \frac{1}{C}\frac{\partial y}{\partial \nu},\ \frac{1}{C}\frac{\partial z}{\partial \nu}$$

are mutually perpendicular, and form a right-handed set of axes, if

$$\begin{vmatrix} \frac{\partial x}{\partial \lambda}, & \frac{\partial y}{\partial \lambda}, & \frac{\partial z}{\partial \lambda} \\ \frac{\partial x}{\partial \mu}, & \frac{\partial y}{\partial \mu}, & \frac{\partial z}{\partial \mu} \\ \frac{\partial x}{\partial \nu}, & \frac{\partial y}{\partial \nu}, & \frac{\partial z}{\partial \nu} \end{vmatrix} = ABC.$$

We shall suppose the signs of A,B,C taken to satisfy the condition. I take the normals to λ = constant, μ = constant, ν = constant at any point as axes of x, y, z and we have to determine the infinitesimal rotations for the set of moving axes coinciding with these.

The straight line $\left(\frac{1}{A}\frac{\partial x}{\partial \lambda}, \frac{1}{B}\frac{\partial x}{\partial \mu}, \frac{1}{C}\frac{\partial x}{\partial \nu}\right)$, and similar lines are fixed in space. Therefore

$$d\left(\frac{1}{A}\frac{\partial x}{\partial \lambda}\right) - \frac{1}{B}\frac{\partial x}{\partial \mu}\partial_3 + \frac{1}{C}\frac{\partial x}{\partial \nu}\partial_2 = 0.$$

Similarly $$d\left(\frac{1}{A}\frac{\partial y}{\partial \lambda}\right) - \frac{1}{B}\frac{\partial y}{\partial \mu}\partial_3 + \frac{1}{C}\frac{\partial y}{\partial \nu}\partial_2 = 0.$$

and $$d\left(\frac{1}{A}\frac{\partial z}{\partial \lambda}\right) - \frac{1}{B}\frac{\partial z}{\partial \mu}\partial_3 + \frac{1}{C}\frac{\partial z}{\partial \nu}\partial_2 = 0.$$

Multiplying these respectively by

$$\frac{\partial y}{\partial \nu}\frac{\partial z}{\partial \lambda} - \frac{\partial y}{\partial \lambda}\frac{\partial z}{\partial \nu},\quad \frac{\partial z}{\partial \nu}\frac{\partial x}{\partial \lambda} - \frac{\partial z}{\partial \lambda}\frac{\partial x}{\partial \nu},\quad \frac{\partial x}{\partial \nu}\frac{\partial y}{\partial \lambda} - \frac{\partial y}{\partial \nu}\frac{\partial x}{\partial \lambda}$$

we get

$$AC\partial_3=\sum d\left(\frac{1}{A}\frac{\partial x}{\partial\lambda}\right)\left(\frac{\partial y}{\partial\nu}\frac{\partial z}{\partial\lambda}-\frac{\partial y}{\partial\lambda}\frac{\partial z}{\partial\nu}\right)$$

Now $$\frac{\frac{1}{B}\frac{\partial x}{\partial\mu}}{\frac{1}{C}\left(\frac{\partial y}{\partial\nu}\frac{\partial z}{\partial\lambda}-\frac{\partial y}{\partial\lambda}\frac{\partial z}{\partial y}\right)}=\text{,,}=\text{,,}=1,$$

multiplying respectively by $\frac{\partial x}{\partial\mu}, \frac{\partial y}{\partial\mu}, \frac{\partial z}{\partial\mu}$ and adding.

$$\text{Thus } \partial_3=\sum d\left(\frac{1}{A}\frac{\partial x}{\partial\lambda}\right)\cdot\frac{1}{B}\frac{\partial x}{\partial\mu}$$

$$=\frac{1}{AB}\sum\left\{\frac{\partial^2 x}{\partial\lambda^2}\frac{\partial x}{\partial\mu}d\lambda+\frac{\partial^2 x}{\partial\lambda\,\partial\mu}\frac{\partial x}{\partial\mu}d\mu+\frac{\partial^2 x}{\partial\lambda\,\partial\nu}\frac{\partial x}{\partial\mu}d\nu\right\}$$

$$-d\left(\frac{1}{A}\right)\cdot\frac{1}{B}\sum\frac{\partial x}{\partial\lambda}\frac{\partial x}{\partial\mu}.$$

The last term is zero and also the co-efficient of $d\nu$, also

$$\sum\frac{\partial^2 x}{\partial\lambda\,\partial\mu}\frac{\partial x}{\partial\mu}=B\frac{\partial B}{\partial\lambda},$$

and $$\sum\frac{\partial^2 x}{\partial\lambda^2}\frac{\partial x}{\partial\mu}=-\sum\frac{\partial x}{\partial\lambda}\frac{\partial^2 x}{\partial\lambda\,\partial\mu},$$

$$=-A\frac{\partial A}{\partial\mu},\text{ since }\sum\frac{\partial x}{\partial\lambda}\frac{\partial x}{\partial\mu}=0,$$

whence $$\partial_3=\frac{1}{A}\frac{\partial B}{\partial\lambda}d\mu-\frac{1}{B}\frac{\partial A}{\partial\mu}$$

Similarly $$\partial_1=\frac{1}{B}\frac{\partial C}{\partial\mu}d\nu-\frac{1}{C}\frac{\partial B}{\partial\nu}d\mu.$$

$$\partial_2=\frac{1}{C}\frac{\partial A}{\partial\nu}d\lambda-\frac{1}{A}\frac{\partial C}{\partial\lambda}d\nu.$$

We can now determine the radii of curvature and torsion of these lines of curvature.

Consider the line for which μ, ν are constant. Then

$$\Delta S^2=A^2 d\lambda^2$$

The tangent being (1,0,0), we have

$$\Delta\varepsilon^2=\partial_3^2+\partial_2^2$$

and $$\frac{1}{\rho^2}=\frac{1}{A^2B^2}\left(\frac{\partial A}{\partial\mu}\right)^2+\frac{1}{A^2C^2}\left(\frac{\partial A}{\partial\nu}\right)^2$$

since $d\mu$, $d\nu$ are both zero.

The consecutive tangent is $(1, \partial_3, -\partial_2)$. The binormal is therefore $(0, \cos \chi, \sin \chi)$.

where
$$\tan \chi = \frac{\partial_3}{\partial_2} = -\frac{1}{B}\frac{\partial A}{\partial \mu} \Big/ \frac{1}{C}\frac{\partial A}{\partial \nu}$$

and
$$\Lambda T^2 = (-\cos \chi\, \partial_3 + \sin \chi\, \partial_2)^2 + (-\sin \chi\, d\chi)^2 + (\cos \chi\, d\chi)^2 = d\chi$$

whence
$$\frac{1}{\sigma} = \frac{1}{A}\frac{\partial \chi}{\partial \lambda}.$$

These results were given in the Mathematical Tripos, Part II, 1903.

§ 35. *On the system of confocal quadrics.*

If we consider the three surfaces
$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1,$$
$$\frac{x^2}{a^2+\mu} + \frac{y^2}{b^2+\mu} + \frac{z^2}{c^2+\mu} = 1,$$
$$\frac{x^2}{a^2+\nu} + \frac{y^2}{b^2+\nu} + \frac{z^2}{c^2+\nu} = 1;$$

we obtain
$$x^2 = \frac{(a^2+\lambda)(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)} \text{ etc.},$$
$$ds^2 = \frac{1}{4}\frac{(\lambda-\mu)(\lambda-\nu)}{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)} d\lambda^2 + +$$

Then
$$A^2 = \frac{(\lambda-\mu)(\mu-\nu)}{4(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)} = \frac{(\lambda-\mu)(\lambda-\nu)}{4\,F(\lambda)} \text{ say,}$$

and
$$B^2 = \frac{(\mu-\lambda)(\mu-\nu)}{4\,F(\mu)},\ C^2 = \frac{(\nu-\lambda)(\nu-\mu)}{4\,F(\nu)}.$$

Therefore
$$\frac{1}{A}\frac{\partial A}{\partial \mu} = \frac{\partial \log A}{\partial \mu} = \frac{1}{2(\mu-\lambda)},\ \frac{1}{A}\frac{\partial A}{\partial \nu} = \frac{1}{2(\nu-\lambda)};$$

whence
$$\frac{1}{\rho^2} = \frac{F(\mu)}{(\mu-\lambda)^2(\mu-\nu)} + \frac{F(\nu)}{(\nu-\lambda)^2(\nu-\mu)}$$
$$= \frac{(\nu-\lambda)^2 F(\mu) - (\mu-\lambda)^2 F(\nu)}{(\mu-\lambda)^2(\nu-\lambda)^2(\mu-\nu)}$$

Again
$$\tan\chi = -\frac{1}{B}\frac{\partial \log A}{\partial \mu} \Big/ \frac{1}{C}\frac{\partial \log A}{\partial \nu}$$
$$= -\frac{\nu-\lambda}{\mu-\lambda}\cdot\left\{-\frac{\nu-\lambda}{\mu-\lambda}\cdot\frac{F(\mu)}{F(\nu)}\right\}^{\frac{1}{2}}$$
$$= -\left(\frac{\nu-\lambda}{\mu-\lambda}\right)^{\frac{3}{2}}\left\{-\frac{F(\mu)}{F(\nu)}\right\}^{\frac{1}{2}};$$

whence $\sec^2\chi\,\dfrac{\partial\chi}{\partial\lambda}=\dfrac{\frac{3}{2}(\mu-\nu)(\nu-\lambda)^{\frac{1}{2}}}{(\mu-\lambda)^{\frac{5}{2}}}\left\{-\dfrac{F(\mu)}{F(\nu)}\right\}^{\frac{1}{2}}$

and
$$\frac{\partial\chi}{\partial\lambda}=\frac{3}{2}\,\frac{(\mu-\nu)\{-(\nu-\lambda)(\mu-\lambda)F(\mu)F(\nu)\}^{\frac{1}{2}}}{(\mu-\lambda)^3F(\nu)-\nu-(\lambda)^3F(\lambda)}.$$

Thus
$$\frac{1}{\sigma}=\frac{1}{A}\frac{\partial\chi}{\partial}\;\frac{3\{-F(\lambda)F(\mu)F(\nu)\}^{\frac{1}{2}}(\mu-\nu)}{(\mu-\lambda)^3F(\nu)-(\nu-\lambda)^3F(\mu)}$$
$$=\frac{3\rho^2\{-F(\lambda)F(\mu)F(\nu)\}^{\frac{1}{2}}}{(\lambda-\mu)^3(\nu-\lambda)^3}$$

by use of the above value of $\dfrac{1}{\rho^2}$.

If ρ_1, σ_1 ; ρ_2, σ_2 ; ρ_3, σ_3 are respectively the radii of curvature and torsion of the three lines of intersection of the confocal surfaces $\lambda, \mu\ \nu$, we have the relations
$$\rho_1^2\sigma_1(\mu-\nu)^3=\rho_2^2\sigma_2(\nu-\lambda)^3=\rho_3^2\sigma_3(\lambda-\mu)^3,$$
a result given in the Mathematical Tripos in 1895.

§ 36. *Geodesics on the ellipsoid* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$.

We put $\lambda=0$; the lines $\mu=$constant, $\nu=$constant, give the lines of curvature.

If $(0, \cos\theta, \sin\theta)$ are the direction cosines of a tangent to a geodesic, the consecutive tangent is
$$-\cos\theta.\partial_3+\sin\theta\partial_2,\ -\sin\theta d\theta-\sin\theta.\partial_1,\ \cos\theta d\theta+\cos\theta.\partial_1.$$

In a geodesic the binormal is $(0,-\sin\theta,\cos\theta)$ and this is perpendicular to the consecutive tangent, if
$$d\theta+\partial_1=0$$
or
$$d\theta+\frac{1}{B}\frac{\partial C}{\partial\mu}d\nu-\frac{1}{C}\frac{\partial B}{\partial\nu}d\mu=0$$
whence
$$ds^2=B^2d\mu^2+C^2d\nu^2,\ \text{gives}\ \tan\theta=\frac{C}{B}\frac{d\nu}{d\mu}.$$
Therefore
$$2\sin\theta\cos\theta d\theta+2\sin^2\theta\frac{\partial\log C}{\partial\mu}d\mu-2\cos^2\theta\frac{\partial\log B}{d\nu}d\nu=0,$$
or
$$2\sin\theta\cos\theta d\theta+\sin^2\theta\frac{d\mu}{\mu-\nu}-\cos^2\theta\frac{d\nu}{\nu-\mu}=0,$$
or
$$2(\mu-\nu)\sin\theta\cos\theta d\theta+\sin^2\theta d\mu+\cos^2\theta d\nu=0.$$
which integrates into
$$\mu\sin^2\theta+\nu\cos^2\theta=\text{constant}=k.$$

This is Joachimsthal's integral of the geodesic lines.

For the radii of curvature and torsion of the geodesic, we have

$$\Delta\varepsilon = -\cos\theta\,\partial_3 + \sin\theta\,\partial_2, \text{ where } \partial_3 = \frac{1}{A}\frac{\partial B}{\partial\lambda}d\mu,\ \partial_2 = -\frac{1}{A}\frac{\partial C}{\partial\lambda}d\nu;$$

whence
$$\frac{1}{\rho} = \cos\theta.\frac{1}{A}\frac{\partial B}{\partial\lambda}\frac{d\mu}{ds} + \sin\theta.\frac{1}{A}\frac{\partial C}{\partial\lambda}\cdot\frac{\partial\nu}{ds}$$
$$= \frac{1}{A}\left\{\cos^2\theta\frac{\partial\log B}{\partial\lambda} + \sin^2\theta\frac{\partial\log C}{\partial\lambda}\right\}$$
$$= \frac{1}{2A}\left\{\frac{\cos^2\theta}{\lambda-\mu} + \frac{\sin^2\theta}{\lambda-\nu}\right\}.$$

From Euler's Theorém we see that $2\,A(\lambda-\mu)$, $2A(\lambda-\nu)$ are the principal radii of curvature of the surface $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1$, at the point where it is met by the confocals whose parameters are μ, ν.

Again
$$\frac{dT}{ds} = \sin\theta\frac{1}{A}\frac{\partial B}{\partial\lambda}\frac{d\mu}{ds} - \cos\theta\frac{1}{A}\frac{\partial C}{\partial\lambda}.\frac{d\nu}{ds}$$
$$= \frac{\sin\theta\cos\theta}{2A}\left(\frac{1}{\lambda-\mu} - \frac{1}{\lambda-\nu}\right).$$

A simple reduction will show that if $(x.y,z)$ is the point of intersection of the confocals λ, μ, ν.

$$x^2+y^2+z^2 = \sum\frac{(a^2+\lambda)(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)} = a^2+b^2+c^2+\lambda+\mu+\nu;$$

and if p is the perpendicular from the centre on the tangent plane at (x,y,z) to the confocal λ

$$\frac{1}{p^2} = \sum\frac{x^2}{(a^2+\lambda)^2} = \sum\frac{(a^2+\mu)(a^2+\nu)}{(a^2+\lambda)(a^2-b^2)(a^2-c^2)} = \frac{(\lambda-\mu)(\lambda-\nu)}{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}.$$

We thus see that $\frac{1}{p} = 2\,A$, and that the principal radii of curvature of the ellipsoid $\lambda =$ constant, are $\frac{\lambda-\mu}{p}, \frac{\lambda-\nu}{p}$ (Frost, §720). The expression for the radius of torsion of a geodesic given in Frost, p. 328, LV ex. (2) is easily obtained.

For,
$$\frac{1}{\sigma^2} = -\frac{(k-\nu)(k-\mu)}{(\lambda-\mu)^3(\lambda-\nu)^3}(a^2+\lambda)(b^2+\lambda)(c^2+\lambda).$$

where $k = \mu \sin^2\theta + \nu\cos^2\theta$.

Putting $\lambda = 0$, so that $\mu+\nu = x^2+y^2+z^2-a^2-b^2-c^2$, $\mu\nu = \frac{a^2b^2c^2}{p^2}$, we obtain the given expression.

§ 37. It will be convenient to change the signs of μ, ν, k so as to have all quantities positive.

We consider therefore the confocals to $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, given by $\frac{x^2}{a^2-\mu}+\frac{y^2}{b^2-\mu}+\frac{z^2}{c^2-\mu}=1$, and $\frac{x^2}{a^2-\nu}+\frac{y^2}{b^2-\nu}+\frac{z^2}{c^2-\nu}=1$, $b^2>\mu>c^2$ and $a^2>\nu>b^2$. We have therefore

$$x^2=\frac{a^2(a^2-\mu)(a^2-\nu)}{(a^2-b^2)(a^2-c^2)},\ y^2=\frac{b^2(b^2-\mu)(\nu-b^2)}{(a^2-b^2)(b^2-c^2)},\ z^2=\frac{c^2(\mu-c^2)(\nu-c^2)}{(a^2-c^2)(b^2-c^2)}$$

The geodesics on the ellipsoid are given by

$$\mu \sin^2\theta+\nu\cos^2\theta=k,$$

whence

$$\tan^2\theta=\frac{\nu-k}{k-\mu}=\frac{\nu(\nu-\mu)}{(a^2-\nu)(b^2-\nu)(c^2-\nu)}\cdot\frac{(a^2-\mu)(b^2-\mu)(c^2-\mu)}{\mu(\mu-\nu)}\left(\frac{d\nu}{d\mu}\right)^2,$$

since $\tan\theta=\frac{C}{B}\cdot\frac{d\nu}{d\mu}$.

Writing $F(\mu)$ for $(a^2-\mu)(b^2-\mu)(c^2-\mu)$ the geodesics are given by

$$d\mu\sqrt{\frac{\mu}{(\mu-k)F(\mu)}}=\pm d\nu\sqrt{\frac{\nu}{(\nu-k)F(\nu)}},$$

where $F(\mu)$ is $-ve$, $F(\nu)$ is $+ve$, and $\nu>k>\mu$.

The integration of this equation gives the family of geodesics for which k is given. The singular solution of this equation is $\nu-k=0$, $\mu-k=0$. Thus the envelope of the geodesics for which k is the same is one or other of the lines of curvature $\nu=k$, $\mu=k$, the former being taken when $k>b^2$ and the latter when $k<b^2$. There are thus three types of geodesics on the ellipsoid :

(1) Those for which $a^2>k>b^2$, when the geodesics wind indefinitely round the ellipsoid touching alternately the two parts of the line of curvature $\nu=k$.

(2) Those for which $b^2>k>c^2$, when the geodesics touch alternately the two parts of the line of curvature $\mu=k$.

(3) Those for which $k=b^2$, in which case the singular solution is $\mu=\nu=b^2$, and this gives the real umbilics. These geodesics, therefore continually pass through two umbilics.

For the arc of the geodesics, writing

$$d\mu\sqrt{\frac{\mu}{(\mu-k)F(\mu)}}=\pm d\nu\sqrt{\frac{\mu}{(\nu-k)F(\nu)}}=du,$$

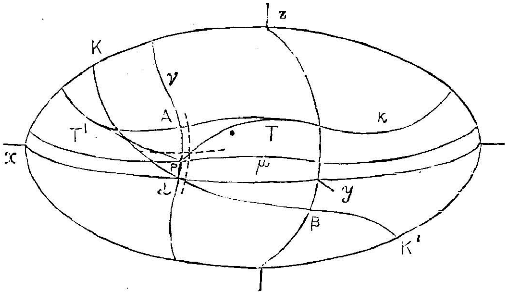
we have $ds^2 = B^2 d\mu^2 + C^2 d\nu^2$

$$= \frac{\mu-\nu}{4}(\mu-k)\,du^2 + \frac{\nu-\mu}{4}(\nu-k)\,du^2;$$

whence $ds = \frac{\nu-\mu}{2} du = \frac{1}{2}\left\{(\nu-k)+(k-\mu)\right\} du$;

$$= \frac{1}{2}\left\{ d\mu \sqrt{\frac{\mu(\mu-k)}{F(\mu)}} \pm d\nu \sqrt{\frac{\nu(\nu-k)}{F(\nu)}} \right\}$$

This assumes $d\mu$ $+$, and $d\nu$ $\pm$ when the upper, or lower sign is taken. These equations contain the whole theory of geodesics on the ellipsoid.



In the figure P is the point (μ, ν) PT, PT′ two geodesic lines for the given value of k; TT′ the line of curvature $\mu = k$, to which the geodesics are tangent. The broken lines indicate the consecutive lines of curvature $\mu + d\mu$, $\nu + d\nu$ for $d\mu$, $d\nu$ positive. That they are correctly placed is shown by considering that $\frac{dz^2}{d\mu} = \frac{c^2(\nu - c^2)}{(a^2-c^2)(b^2-c^2)}$ is positive and thus z^2 increases with μ.

So $\frac{dx^2}{d\nu}$ is negative and thus x^2 diminishes as ν increases.

Since
$$\tan\theta = \pm\sqrt{\frac{\nu-k}{k-\mu}} = \pm\frac{C\,d\nu}{B\,d\mu},$$

we see that for PT, $\tan\theta = +\sqrt{\frac{\nu-k}{k-\mu}}$ and for PT′, $\tan\theta = -\sqrt{\frac{\partial-\nu}{\nu-k}}$.

Thus PT, PT′ are equally inclined to the lines of curvature through P. Also for PT, PT′ $d\mu$ is positive, and $d\nu$ positive & negative respectively.

Hence $$arc\ \mathrm{PT} = \tfrac{1}{2}\int_{\mu}^{k}\sqrt{\frac{\mu(\mu-k)}{F(\mu)}}d\mu + \tfrac{1}{2}\int_{\nu}^{\nu'_0}\sqrt{\frac{\nu(\nu-k)}{F(\nu)}}d\nu$$

$$arc\ \mathrm{PT}' = \tfrac{1}{2}\int_{\mu}^{k}\sqrt{\frac{\mu(\mu-k)}{F(\mu)}}d\mu - \tfrac{1}{2}\int_{\nu}^{\nu'_0}\sqrt{\frac{\nu(\nu-k)}{F(\nu)}}d\nu,$$

where ν_0, ν_0' are the values of the parameters at T, T′ respectively.

Adding

$$arc\ (\mathrm{PT}+\mathrm{PT}') = \int_{\mu}^{k}\sqrt{\frac{\mu(\mu-k}{F(\mu)}}d\mu + \tfrac{1}{2}\int_{\nu_0'}^{\nu_0}\sqrt{\frac{\nu(\nu-k)}{F(\nu)}}d\nu.$$

Now for the line of curvature $\mu = k =$ constant, $ds = Cd\nu$; whence

$$arc\ \mathrm{TT}' = \tfrac{1}{2}\int_{\nu_0'}^{\nu_0}\sqrt{\frac{\nu(\nu-k)}{F(\nu)}}d\nu.$$

Thus finally we have Chasle's theorem

$$arc\ (\mathrm{PT}+\mathrm{PT}'-\mathrm{TT}') = \int_{\mu}^{k}\sqrt{\frac{\mu(\mu-k)}{F(\mu)}}d\mu = \text{constant},$$

if P moves along the line of curvature μ.

Again $$\mathrm{PT}-\mathrm{PT}' = \tfrac{1}{2}\int_{\nu}^{\nu_0}d\nu\sqrt{\frac{\nu(\nu-k)}{F(\nu)}} - \tfrac{1}{2}\int_{\nu_0'}^{\nu}d\nu\sqrt{\frac{\nu(\nu-k)}{F(\nu)}}$$

$$= \mathrm{AT}-\mathrm{AT}';$$

or the difference of the arcs PT and PT′ is constant, if P moves along line of curvature $\nu =$ constant.

§ 38 To further illustrate these formulae and the ambiguities of sign connected with them, I will calculate the length of the geodesic joining two umbilics.

Let K$\alpha\beta$K′ be such a geodesic, where K, K′ are opposite umbilics; and α, β the points where the geodesic crosses the planes $z=0$, $x=0$.

For such a geodesic $k=b^2$, and at the umbilics $\mu=\nu=b^2$.

$$\tan\theta = \pm\sqrt{\frac{\nu-b^2}{b^2-\mu}}.$$

$$\frac{d\mu}{b^2-\mu}\sqrt{\frac{\mu}{(a^2-\mu)(\mu-c^2)}} = \pm\frac{d\nu}{\nu-b^2}\sqrt{\frac{\nu}{(a^2-\nu)(\nu-c^2)}}$$

$$ds = \tfrac{1}{2}d\mu\sqrt{\frac{\mu}{(a^2-\mu)(\mu-c^2)}} \pm \tfrac{1}{2}d\nu\sqrt{\frac{\nu}{(a^2-\nu)(\nu-c^3)}}$$

From α to K, $d\mu$ is positive, $d\nu$ is negative, therefore

$$arc\ \alpha\mathrm{K}=\tfrac{1}{2}\int_{c^2}^{b^2} d\mu\sqrt{\frac{\mu}{(a^2-\mu)(\mu-c^2)}}-\tfrac{1}{2}\int_{\nu}^{a^2} d\nu\sqrt{\frac{\nu}{(a^2-\nu)(\nu-c^2)}}$$

From α to β, $d\mu$ is positive, and $d\nu$ is positive, therefore

$$arc\ \alpha\beta=\tfrac{1}{2}\int_{c^2}^{\mu_1} d\mu\sqrt{\frac{\mu}{(a^2-\mu)(\mu-c^2)}}+\tfrac{1}{2}\int_{\nu}^{a^2} d\nu\sqrt{\frac{\nu}{(a^2-\nu)(\nu-c^2)}}$$

From β to K′ $d\mu$ is positive, $d\nu$ is negative, therefore

$$arc\ \beta\mathrm{K}'=\tfrac{1}{2}\int_{\mu_1}^{b^2} d\mu\sqrt{\frac{\mu}{(a^2-\mu)(\mu-c^2)}}-\tfrac{1}{2}\int_{a^2} d\nu\sqrt{\frac{\nu}{(a^2-\nu)(\nu-c^2)}},$$

the limits being determined by $\mu=c^2$ at α, and $\nu=a^2$ at β.

Adding these three arcs we get

$$arc\ \mathrm{KK}'=\int_{c^2}^{b^2} d\mu\sqrt{\frac{\mu}{(a^2-\mu)(\mu-c^2)}}+\int_{b^2}^{a^2} d\nu\sqrt{\frac{\nu}{(a^2-\nu)(\nu-c^2)}}$$

$$=\int_{c^2}^{b^2} d\mu\sqrt{\frac{\mu}{(a^2-\mu)(\mu-c^2)}}$$

$$=2\int_0^{\frac{\pi}{2}}(a^2\cos^2\theta+c^2\sin^2\theta)^{\frac{1}{2}}d\theta,\ \text{writing}\ \mu=a^2\cos^2\theta+c^2\sin^2\theta,$$

=half the circumference of the section of the ellipsoid by the plane $y=0$.

§ 39. To determine the principal radii of curvature of the ellipsoid at the point μ, ν; or the radii of curvature of the normal sections at P containing the tangents to the lines of curvature at P.

Consider ν=constant. The consecutive normal to the surface is

$$\frac{x}{1}=\frac{y-\mathrm{B}\,d\mu}{\partial_3}=\frac{z}{-\partial_2}\cdot\ \text{where}\ \nu\ \text{is constant.}$$

$$\therefore\quad d\,\varepsilon^2=\partial_3^2+\partial_2^2=\frac{1}{\mathrm{A}^2}\left(\frac{\partial\mathrm{B}}{\partial\lambda}\right)^2 d\mu^2,\ \text{and}\ ds^2=\mathrm{B}^2d\mu^2.$$

$$\therefore\quad \frac{1}{\mathrm{R}'^2}=\frac{1}{\mathrm{A}^2}\left(\frac{\partial\log\mathrm{B}}{\partial\lambda}\right)^2=\frac{4\,a^2b^2c^2}{\mu\nu}\cdot\frac{1}{4}\cdot\frac{1}{\mu^2}=\frac{a^2b^2c^2}{\mu^3\nu}$$

$$\therefore\quad \mathrm{R}'=\frac{\mu^{\frac{3}{2}}\nu^{\frac{1}{2}}}{abc},\ \mathrm{R}=\frac{\mu^{\frac{1}{2}}\nu^{\frac{3}{2}}}{abc}$$

We may write these $R=\frac{R'^3 a^2b^2c^2}{\mu^4}$, $R'=\frac{R^3a^2b^2c^2}{\nu^4}$; so that R the radius of curvature of the normal section along $\mu=$constant, varies as R'^3, the radius of curvature of the perpendicular normal section being R′.

The specific curvature $RR'=\frac{\mu^2\nu^2}{a^2b^2c^2}=\frac{a^2b^2c^2}{p^4}$, where p is the perpendicular from the centre of the ellipsoid on the tangent plane at (μ,ν).

The locus of points for which RR′ or $\mu\nu$ or p is constant is the curve called the *polhode.* (Routh: *Rigid Dynamics,* Vol. II, Ex. 1., p. 97.)

§ 40. I propose to obtain the equation of the *herpolhode* in the form given in the Mathematical Tripos of 1893.

The herpolhode is the locus of the point of contact of an ellipsoid whose centre is fixed, that rolls on a fixed plane.

The tangent at each instant coincides with the tangent at the corresponding point of the polhode.

With $\lambda=0$, we have for the perpendicular from the centre on the tangent plane at (μ,ν), $\frac{1}{p^2}=\frac{\mu\nu}{a^2b^2c^2}$.

If (x,y,z) is the point on the ellipsoid, the foot of the perpendicular from the centre is $(p^2x/a^2,\ p^2y/b^2,\ p^2z/c^2)$.

The point (X, Y, Z) referred to the principal axes of the ellipsoid becomes when referred to the moving axis at (μ,ν), the axis of y being normal to $\mu=$constant, and the axis of z normal to $\nu=$constant:

$$(X-x)\frac{1}{A}\frac{\partial x}{\partial\lambda}+(Y-y)\frac{1}{A}\frac{\partial y}{\partial\lambda}+(Z-z)\frac{1}{A}\frac{\partial z}{\partial\lambda},$$

$$(X-x)\frac{1}{B}\frac{\partial x}{\partial\mu}+(Y-y)\frac{1}{B}\frac{\partial y}{\partial\mu}+(Z-z)\frac{1}{B}\frac{\partial z}{\partial\mu},$$

$$(X-x)\frac{1}{C}\frac{\partial x}{\partial\nu}+(Y-y)\frac{1}{C}\frac{\partial y}{\partial\nu}+(Z-z)\frac{1}{C}\frac{\partial z}{\partial\nu}.$$

Now $$x^2+y^2+z^2=a^2+b^2+c^2+\lambda+\mu+\nu.$$

Therefore $$x\frac{\partial x}{\partial\lambda}+y\frac{\partial y}{\partial\lambda}+z\frac{\partial z}{\partial\lambda}=\frac{1}{2}.$$

Also $$X\frac{\partial x}{\partial\lambda}+Y\frac{\partial y}{\partial\lambda}+Z\frac{\partial z}{\partial\lambda}=p^2\left\{\frac{x}{a^2}\frac{\partial x}{\partial\lambda}+\frac{y}{b^2}\frac{\partial y}{\partial\lambda}+\frac{z}{c^2}\frac{\partial z}{\partial\lambda}\right\}$$

$$=\frac{1}{2}p^2\sum\frac{(a^2+\mu)(a^2+\nu)}{a^2(a^2-b^2)(a^2-c^2)}=\frac{1}{2}p^2\frac{\mu\nu}{a^2b^2c^2}=\frac{1}{2}.$$

So $$X\frac{\partial x}{\partial\mu}+Y\frac{\partial y}{\partial\mu}+Z\frac{\partial z}{\partial\mu}=\frac{1}{2}p^2\sum\frac{1}{a^2}\cdot\frac{a^2(a^2+\nu)}{(a^2-b^2)(a^2-c^2)}=0,$$

and $$X\frac{\partial x}{\partial\nu}+Y\frac{\partial y}{\partial\nu}+Z\frac{\partial z}{\partial\nu}=0,$$

these latter results being of course geometrically obvious.

Thus the coordinates of the foot of the perpendicular from the centre of the ellipsoid on the tangent plane at $(0, \mu, \nu)$ are

$$\left(0, -\frac{1}{2B}, -\frac{1}{2C}\right)$$

referred to the moving axis at $(0, -\mu, \nu)$.

The direction cosines of the tangent to the polhode, $\mu\nu =$constant, are proportional to, $(0, Bd\mu, Cd\nu)$, where $\mu d\nu + \nu d\mu = 0$. Thus the direction cosines of the tangent are proportional to $(0, B\mu, -C\nu)$.

If now r is the radius vector from O, the foot of the perpendicular from the centre of the ellipsoid on the fixed tangent plane, and p the perpendicular on the tangent to the herpolhode from O, and θ the inclination of this tangent to the radius vector, we have

$$r^2 - p^2 = r^2 \cos^2\theta = r^2 \frac{(\mu-\nu)^2}{\left(\frac{1}{B^2}+\frac{1}{C^2}\right)(B^2\mu^2+C^2\nu^2)},$$

since θ is the angle between the lines whose direction cosines are proportional to $\left(0, -\frac{1}{2B}, -\frac{1}{2C}\right)$; $(0, B\mu, -C\nu)$. Also

$$r^2 = \frac{1}{4}\left(\frac{1}{B^2}+\frac{1}{C^2}\right) = \frac{\nu(a^2+\mu)(b^2+\mu)(c^2+\mu) - \mu(a^2+\nu)(b^2+\nu)(c^2+\nu)}{\mu\nu(\mu-\nu)}$$

$$= a^2+b^2+c^2+\mu+\nu - \frac{a^2b^2c^2}{\mu\nu}.$$

as can easily be seen from geometrical conditions.

Also, if we write $\mu\nu = \frac{a^2b^2c^2}{K^2}$, we can obtain the equation of the herpolhode by eliminating μ, ν. We have in fact

$$\frac{1}{r^2-p^2} = \frac{4(B^2\mu^2+C^2\nu^2)}{(\mu-\nu)^2}$$

$$= \left[\frac{\mu^3}{(a^2+\mu)(b^2+\mu)(c^2+\mu)} - \frac{\nu^3}{(a^2+\nu)(b^2+\nu)(c^2+\nu)}\right]\frac{1}{\mu-\nu}$$

$$= \sum_{a^2b^2c^2} \frac{a^6}{(a^2-b^2)(a^2-c^2)(a^2+\mu)(a^2+\nu)}, \text{ by partial fractions.}$$

$$= \sum \frac{a^6}{(a^2-b^2)(a^2-c^2)\left\{a^4 + a^2(r^2-a^2-b^2-c^2+k^2) + \frac{a^2b^2c^2}{k^2}\right\}}$$

whence on reduction, we can write the equation,

$$-\frac{(b^2-c^2)(c^2-a^2)(a^2-b^2)}{r^2-p^2} = \sum \frac{a^4(b^2-c^2)}{r^2 + \frac{(K^2-b^2)(K^2-c^2)}{K^2}}$$

agreeing with the result given in the Mathematical Tripos.

SHORT NOTES:

Note on Mersenne's Number $2^{71}-1$.

It was announced in 'Nature' dated 12th August 1909 that Lieut. Col. Cunningham found that

$$2^{71}-1=228479\times 10334355636337793.$$

the nature of the larger factor not being known.

I have now found that

$$10334355636337793=48544121\times 212885833.$$

6th March, 1912. V. RAMESAM.

The Volume of a Tetrahedron in terms of its six edges.

1. Before applying our method to a tetrahedron, let us consider the case of a triangle ABC, whose vertices are (x_1, y_1) (x_2, y_2) (x_3, y_3).

The determinant

$$\begin{vmatrix} x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

is readily seen to be equal to $-2\triangle$, where $\triangle$ denotes the area of the triangle.

Again

$$8\triangle=\begin{vmatrix} 1 & -2x_1 & -2y_1 & x_1^2+y_1^2 \\ 1 & -2x_2 & -2y_2 & x_2^2+y_2^2 \\ 1 & -2x_3 & -2y_3 & x_3^2+y_3^2 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Multiplying the above determinants together, we get

$$-16\triangle^2=\begin{vmatrix} 0 & c^2 & b^2 & 1 \\ c^2 & 0 & a^2 & 1 \\ b^2 & a^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

a, b, c standing for the sides of the triangle.

That is

$$16\,\Delta^2=2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4.$$

2. We can, similarly, obtain the relation among the six lines joining any four coplanar points.

For, if (x_4y_4) is a fourth point, whose distances from A,B,C are p,q,r, the product of the matrices

$$\begin{vmatrix} x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \\ x_4^2+y_4^2 & x_4 & y_4 & 1 \\ 1 & 0 & 0 & 0 \end{vmatrix} \qquad \begin{vmatrix} 1 & -2x_1 & -2y_1 & x_1^2+y_1^2 \\ 1 & -2x_2 & -2y_2 & x_2^2+y_2^2 \\ 1 & -2x_3 & -2y_3 & x_3^2+y_3^2 \\ 1 & -2x_4 & -2y_4 & x_4^2+y_4^2 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

must vanish.

Hence

$$\begin{vmatrix} 0 & c^2 & b^2 & p^2 & 1 \\ c^2 & 0 & a^2 & q^2 & 1 \\ b^2 & a^2 & 0 & r^2 & 1 \\ p^2 & q^2 & r^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}=0,$$

which is the relation required.

3. Consider next, a tetrahedron ABCD whose vertices are $(x_1y_1z_1)$ &c., and volume is V.

We know

$$6V=\begin{vmatrix} \Sigma(x_1^2) & x_1 & y_1 & z_1 & 1 \\ \Sigma(x_2^2) & x_2 & y_2 & z_2 & 1 \\ \Sigma(x_3^2) & x_3 & y_3 & z_3 & 1 \\ \Sigma(x_4)^2 & x_4 & y_4 & z_4 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix}.$$

Also

$$48V = \begin{vmatrix} 1 & -2x_1 & -2y_1 & -2z_1 & \Sigma(x_1)^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

Multiplying

$$288V^2 = \begin{vmatrix} 0 & c^2 & b^2 & p^2 & 1 \\ c^2 & 0 & a^2 & q^2 & 1 \\ b^2 & a^2 & 0 & r^2 & 1 \\ p^2 & q^2 & r^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}$$

4. If E is any fifth point, whose distances from A, B, C, D are α, β, γ, δ respectively, we get, as in § 2, the relation

$$\begin{vmatrix} 0 & c^2 & b^2 & p^2 & \alpha^2 & 1 \\ c^2 & 0 & a^2 & q^2 & \beta^2 & 1 \\ b^2 & a^2 & 0 & r^2 & \gamma^2 & 1 \\ p^2 & q^2 & r^2 & 0 & \delta^2 & 1 \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0,$$

connecting the mutual distances of five points in space.

5. For the radius R of the circumsphere of ABCD, let E coincide with the centre of the sphere; and we have to put $\alpha = \beta = \gamma = \delta = R$ in the above determinant. Solving for R^2, we get, after simplification,

$$36V^2R^2 = \sigma(\sigma - ap)(\sigma - bq)(\sigma - cr)$$

where $2\sigma = ap + bq + cr$. [*vide*: Salmon, § 55.]

17th January 1912. J. C. SWAMINARAYAN.

On Question 330 of Prof. Sanjana.

1. Prof. Sanjana remarks that it is not easy to evaluate the series

$$\frac{1}{1^n}+\frac{1\cdot1}{2\cdot3^n}+\frac{1\cdot3}{2\cdot4}\frac{1}{5^n}+\frac{1\cdot3\cdot5}{2\cdot4\,6}\frac{1}{7^n}+\ldots \text{ ad inf.},$$

if $n>3$. In attempting to sum the series for *all* values of n, I have arrived at the following results:

Let $$f(p)=\frac{1}{1+p}+\frac{1}{2}\frac{1}{3+p}+\frac{1\cdot3}{2\cdot4}\frac{1}{5+p}+\ \ldots$$

$$=\int_0^1 x^p\left(1+\frac{1}{2}x^2+\frac{1\cdot3}{2\cdot4}x^4+\ \ldots\ldots\right)dx$$

$$=\int_0^1\frac{x^p}{\sqrt{1-x^2}}\,dx=\tfrac{1}{2}\int_0^1 x^{\frac{p-1}{2}}(1-x)^{-\frac{1}{2}}dx$$

$$=\tfrac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)}=\frac{\pi^{\frac{1}{2}}}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)}$$

But $\Gamma\left(\frac{p+1}{2}\right)=\frac{\pi^{\frac{1}{2}}}{2^p}\frac{\Gamma(p+1)}{\Gamma\left(\frac{p+2}{2}\right)}$ (*vide*: Williamson, *Integ.*, p. 164.)

$$\therefore\qquad f(p)=\frac{\pi}{2^{p+1}}\frac{\Gamma(p+1}{\left\{\Gamma\left(\frac{p+2}{2}\right)\right\}^2}.$$

$$\therefore \log\{f(p)\}=\log(\pi/2)-p\log 2$$
$$+\frac{p^2}{2}\left(1-\frac{1}{2}\right)S_2-\frac{p^3}{3}\left(1-\frac{1}{2^2}\right)S_3+\quad\ldots\quad(1)$$

where $S_n\equiv\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\ \ldots$ *ad inf.*, [*vide*: Carr's *Synopsis*, 2295].

Again, by expanding $f(p)$ in ascending powers of p, we have

$$f(p)=\left(1+\frac{1}{2}\frac{1}{3}+\frac{1.3}{2\cdot4}\frac{1}{5}+\ \ldots\ \right)-p\left(1+\frac{1}{2}\frac{1}{3^2}+\frac{1.3}{2.4}\frac{1}{5^2}+\ldots\right)$$
$$+p^2\left(1+\frac{1}{2}\frac{1}{3^3}+\frac{1.3}{2.4}\frac{1}{5^3}+\ldots\right)-\ldots$$

$$=\frac{\pi}{2}\{\phi(0)-p.\phi(1)+p^2.\phi(2)-p^3\phi(3)+\ \ldots\}$$

where

$$\frac{1}{1^{n+1}}+\frac{1}{2}\frac{1}{3^{n+1}}+\frac{1.3}{2\cdot4}\frac{1}{5^{n+1}}+\ \ldots\equiv\frac{\pi}{2}\phi(n).$$

Hence (1) may be written

$$\log \pi/2+\log\{\phi(0)-p.\phi(1)+p^2.\phi(2)-\ldots\}$$

$$=\log(\pi/2)-p\log(2)+\frac{p^2}{2}\left(1-\frac{1}{2}\right)S_2-\frac{p^3}{3}\left(1-\frac{1}{2^2}\right)S_3+\ldots$$

$$=\log(\pi/2)-p\sigma_1+\frac{p^2}{2}\sigma_2-\frac{p^3}{3}\sigma_3+\ \ldots$$

where $$\sigma_n\equiv 1-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\ \ldots$$

Differentiating with respect to p, and equating the coefficients of p^{n-1}, we have

$$n\phi(n)\equiv\sigma_1\phi(n-1)+\sigma_2\phi(n-2)+\sigma_3\phi(n-3)+\ldots\text{to } n \text{ terms.}$$

Thus we see that

$$\frac{\pi}{2}.\phi(0)\equiv 1+\frac{1}{2}\frac{1}{3}+\frac{1.3}{2.4}\frac{1}{5}+\ \ldots\ =\frac{\pi}{2}$$

$$\frac{\pi}{2}.\phi(1)\equiv 1+\frac{1}{2}\frac{1}{3^2}+\frac{1\cdot3}{2\cdot4}\frac{1}{5^2}+\ \ldots\ =\frac{\pi}{2}\log 2.$$

$$\frac{\pi}{2}.\phi(2)\equiv 1+\frac{1}{2}\frac{1}{3^3}+\frac{1\cdot3}{2\cdot4}\frac{1}{5^3}+\ \ldots\ =\frac{\pi^3}{48}+\frac{\pi}{4}(\log 2)^2$$

$$\frac{\pi}{2}.\phi(3)\equiv 1+\frac{1}{2}\frac{1}{3^4}+\frac{1\cdot3}{2\cdot4}\frac{1}{5^4}+\ \ldots\ =\frac{\pi^3}{48}\log 2+\frac{\pi^3}{12}(\log 2)^3+\frac{\pi}{6}\sigma_3.$$

$$=\frac{\pi^3}{48}\log 2+\frac{\pi}{12}(\log 2)^3+\frac{\pi}{8}S_3,$$

and so on.

2. More generally, consider the series

$$\frac{1}{b^n}-\frac{a}{\lfloor 1}\frac{1}{(b+1)^n}+\frac{a(a-1)}{\lfloor 2}\frac{1}{(b+2)^n}-\ \ldots,\ldots$$

Writing $\dfrac{\Gamma(b).\Gamma(a+1)}{\Gamma(a+b+1)}\phi(n-1)$ for this, and taking the identity

$$\frac{1}{b+p}-\frac{a}{\lfloor 1}\frac{1}{b+1+p}+\frac{a(a-1)}{\lfloor 2}\frac{1}{b+2+p}-\ \ldots$$

$$=\int_0^1 x^{b+p-1}(1-x)^a dx=\frac{\Gamma(b+p).\Gamma(a+1)}{\Gamma(a+b+p+1)},$$

we find

$$n\phi(n)=\sigma_1\phi(n-1)+\sigma_2\phi(n-2)+\sigma_3\phi(n-3)+\ \ldots\ldots \text{ to } n \text{ terms,}$$

where $\sigma^n\equiv\dfrac{1}{b_n}-\dfrac{1}{(a+b+1)^n}+\dfrac{1}{(b+1)^n}-\dfrac{1}{(a+b+2)_n}+\ \ldots\ .$

Examples: Put $a=-\frac{1}{2}$, $b=\frac{1}{4}$. Then we see that

(i) $$1+\frac{1}{2}\frac{1}{5}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{9}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{13}+\dots=\frac{\left\{\Gamma\left(\frac{1}{4}\right)\right\}^2}{4\sqrt{2\pi}}$$

(ii) $$1+\frac{1}{2}\cdot\frac{1}{5^2}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{9^2}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{13^2}+\dots=\frac{\left\{\Gamma\left(\frac{1}{4}\right)\right\}^2}{4\sqrt{2\pi}}\frac{\pi}{4}.$$

(iii) $$1+\frac{1}{2}\cdot\frac{1}{5^3}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{9^3}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{13^3}+\dots=\frac{\left\{\Gamma\left(\frac{1}{4}\right)\right\}^2}{4^3\sqrt{2\pi}}\left[\frac{\pi^2}{32}+\tfrac{1}{2}S_2'\right]$$

(iv) $$1+\frac{1}{2}\cdot\frac{1}{5^4}+\frac{1.3}{2.4}\frac{1}{9^4}+\frac{1.3.5}{2.4.6}\frac{1}{13^4}+\dots=\frac{\left\{\Gamma\left(\frac{1}{4}\right)\right\}^2}{4\sqrt{2\pi}}\left\{\frac{5\pi^3}{384}+\frac{\pi}{8}S_2'\right\}.$$

where $S_2'=\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\dots$

14th February 1912. S. Ramanujan.

On the formula $\rho=r\frac{dr}{dp}$.

We have

$$r^2=x^2+y^2$$
$$\therefore\quad rdr=xdx+ydy$$
$$=(x\cos\psi+y\sin\psi)ds,\text{ since }\cos\psi=\frac{dx}{ds},\ \&c.$$
$$=\frac{dp}{d\psi}.\,ds,\text{ since }p=x\sin\psi-y\cos\psi,\ \&c.$$

Hence $$\rho=\frac{ds}{d\psi}=r\frac{dr}{dp}.$$

10th September 1911. R. Srinivasan.

The Face of the Sky for May and June 1912.

Sidereal time at 8 p.m.

			May H.	M.	S.		June H.	M.	S.
1	...	...	10	36	39	...	12	38	52
8	...	...	11	4	15	...	13	6	28
15	...	...	11	31	51	...	13	34	4
22	...	...	11	59	27	...	14	1	40
29	...	...	12	27	3	...	14	29	16

From this table the constellations visible during the evenings of May and June can be ascertained by a reference to their position as given in the star charts.

Phases of the Moon.

	May.	H.	M.		June.	H.	M.	
Full Moon	1	3	49	P.M.	...		......	
Last Quarter	9	3	26	„	... 8	8	6	A.M.
New Moon	17	3	47	„	... 15	0	54	P.M.
First Quarter... ...	23	7	41	„	... 22	2	9	A.M.
Full Moon	31	5	0	A.M.	... 29	7	4	P.M.

The Sun

Enters the sign cancer on June 22 at 0·47 A.M.

The Planets.

Mercury begins as morning star. It attains its greatest elongation (26° 3′ W) on May 13, and is in superior conjunction with the Sun on June 17 when it becomes an evening star. It is in conjunction with the Moon on May 15 at 6-15 A.M. and on June 15 and with Saturn on June 3 and with Venus on June 12.

Venus is still a morning star. It is in conjunction with Saturn on May 28 and with the Moon on May 16 and on June 5.

Mars is an evening star. It is in conjunction with the Moon on May 21 and on June 18 at 4-24 P.M.

Jupiter is in opposition to the Sun on June 1; and so, well situated for observation in the evenings during these months. It is in conjunction with the Moon on May 4 at 7·39 A.M., on May 31 at 9-19 A.M. and on June 27 at 9·29 A.M.

Saturn is in conjunction with the Sun on May 14; with the Moon on May 17 and June 13.

Uranus is in conjunction with the Moon on May 8 and Neptune on May 20.

18*th March* 1912.

V. Ramesam.

CORRESPONDENCE :

BRUXELLES, *26th Novembre 1911.*
154 rue verte.

MONSIEUR,

Je reçois par l'intermédiaire de M. Hermann, éditeur à Paris le No d'Octobre 1911 de votre estimable "Journal of the Indian Mathematical Society." J'y treuve une analyse de mon livre "Arithmetique générale" assez peu flatteuse. Permettez-moi de rectifier quelques appréciations inexactes formulées par Mr. Bramg.

1. Le critique eût préfere "Algébre universelle" à "Arithmétique générale" comme titre du livre. J'ignore quelle est la terminologie employée en Angleterre. Mais en Europe, tout le monde appelle *Algebre*, l'ensemble des théories relatives aux fonctions et aux équations. L'expression que j'ai choisie est donc la seule á employer, et non du tout synonyme de "Algébre universelle." Elle a d'ailleurs été déja utilisée par Newton et récemment par Stolz et Couturat.

2. Le critique prétend que je ne définis pas le Nombre. S'il veut bien lire la page VII de l'avant-propos, ainsi que les pages 6, 16, 33, 40, etc...peut-être changera-t-il d'avis.

3. La somme de deux ternions est un ternion et non un quaternion (No. 831).

4. Je n'ai pas "inventé" le ternions pour combler un vide entre les nombres binaires et les quaternions. Le ternion s'impose de luimême. C'est le rapport d'un vecteur quelconque au vecteur-étalon $\bar{r}$ du systéme trirectangulaire de repère (No. 826).

Si les ternions ne sont pas trés utiles, cela ne dépend pas de moi, par conséquent.

5. Le critique estime qu'au lieu des ternions j'aurais dû exposer un résumé de la théorie des *vecteurs* (quaternions sans scalar). Je me permettrai de lui faire remarquer que ces "vecteurs" rentrent dans le cas général des quaternions : ce sont des quaternions dont l'argument est $\frac{\pi}{2}$. *Je les ai traite's tre's longuement* : tout le chapitre IV 4e partie, pp. 221 et suivants, leur est consacré, et il en est souvent encore question dans les chapitres suivants.

6. La petite digression sur $\sqrt{-1}$ est tout à fait hors de propos; je ne m'en suis pas servi, et le critique a l'air d' insinuer le contraire.

7. Je n'ai jamais changé de terminologie. Quant aux noms que j'ai adoptés, je les ai presque tous empruntés á des auteurs estimés J'ai toujours *justifie'* mon choix, et je ne vois pas bien ce qui autorise M. Bramg á parler de "unnecessary nicknames" et de ma passion "medieval" pour une nomenclature speciale. Lorsqu'une notion nouvelle apparait, il faut bien la baptiser si l'ou veut simplifier le discours.

Quaternion et scalar n'avaient jamais été, employés avant Hamilton.

8. Si J'ai substitué i, j, r á i, j, k, cela tient á deux motifs: l'axe r est b'ase de *r*epèr de tout le système; c'est aussi l'axe qui porte les vecteurs dont les mesures sont des nombres *r*éels. D'autre part, k est, employé constamment dans l'expression $2k\pi+\theta$; et comme i, j, r doivent souvent venir 'en exposants, ainsi que $2k\pi+\theta$, il y aurait en deux k ayant une signification différente dans une même formule.

Que Mr. Bramg réserve donc sa compassion pour ce "poor k."

9. Le critique me reproche de ne pas avoir traité des bi-et tri-quaternions. Lorsque j'ai écrit ma préface, je n'en connaissais que ce qu'en dit *Hamilton*: un biquaternion est une expression $q_1+q_2\sqrt{-1}$, dans la quelle q_1 et q_2 sont deux quaternions. Ayant pu définir toutes les mesures de *grandeurs geometriques* sans faire appel autre chose que les quaternions, jétais fondé à dire que les biquaternions étaient pour moi sans emploi.

Mr. Bramg ne connaît pas plus les tri-quaternions que moi il y a un an. J'ai le plaisir de lui signaler l'ouvrage suivant :

Calcul des triquaternions, par *G. Combebiac*; Paris, Gauthier-Villars, 1902. Lorsque mon livre était presque imprimé, J'ai compris en lisant l'ouvrage ci-dessus que les biet triquaternions ne sont pas ce que Hamilton avait imaginé, et qu'ils servent á mesurer des *de'placements*; des *mouvements* et non des grandeurs géométriques, J'ai donc inséré page 269, une petite rectification : j'ai dit : *ce ne sont pas des nombres.* Leur théorie appartient à un cours de Mécanique et non à l'arithmétique (tout comme les vitesses et accelerations.)

10. Le critique ajoute encore que le sujet n'a pas été exposé à fond. Je lui ferai observer que j'ai annoncé dans mon avant-propos, mon intention de *definir* les nombres et d'en donner les *principales proprietes.* Je n'avais donc pas à les étudier dans toutes leurs *applications.* C'est làl'objet d'ouvrages spéciaux pour chaque classe de nombres.

En définitive, Mr. Bramg a lu bien légèrement mon livre; il n'en à pas vu la caractéristique essentielle, et son compte-rendu s'es attardé a discuter des *mots* et non des *idées*.

J'ose espérer, Monsieur, que vous voudrez bien insérer dans l'un de vos prochains numéros les rectifications qui précèdent, aux allégations inexactes de Mr. Bramg, et avec mes remcrcîments anticipés, je vous présente l'assurance de ma consideration distinguée.

PROF. E. DUMONT,
Officier du génie.

The following is an English Translation of the above.

I received through Mr. Hermann, Editor at Paris, the 1911, October issue of your esteemed "Journal of the Indian Mathematical Society." I find a review of my book "Arithmetique Generale", not very flattering. Permit me to rectify some inaccurate remarks made by Mr. Bramg.

1. The critic would have preferred "Algábre Universelle" instead of "Arithmétique Générale" for the title of my book. I do not know the terminology employed in England. But in Europe every oné calls "Algebre" the theories related to the functions and equations. The expression which I have selected is the only one to be employed and is not at all synonimous with Universal Algebra. It has already been used by Newton, and recently by Stolz and Couterat.

2. The critic pretends that I do not define Number. If he reads page VII of the Preface, as well as pages 6, 16, 33, 40, &c., he will change his opinion.

3. The sum of two ternions is a ternion and not a quaternion (No. 831).

4. I did not "invent," the ternion to fill up the gap between the binaries and quaternions. The ternion stands by itself. It is the relation of any vector to the standard vector r of the trirectangular system of reference (No. 826). If the ternions are not useful, I am not responsible.

5. The critic considers that instead of ternions I should have given ia summary of the theory of vectors, (quaternions without scalar), I should be allowed to bring to his notice that the vectors are included n the genéral case of quaternions. They are quaternions whose argument is $\pi/2$. *I have treated them at length*; the whole of Chapter IV, 4th part, pp. 221, &c., are devoted to them, and they are often mentioned in the subsequent chapters.

6. The little digression on $\sqrt{-1}$ is altogether out of place: I have not used it, and the critic seems to insinuate the contrary.

7. I have not changed the terminology. As for the names I have adopted, they have all been taken from well-known authors. I have always justified my choice, and I do not see very well what authorises Mr. Bramg to speak of "unnecessary nicknames", and of my passion "mediæval" for a special nomenclature. When a new notion appears, it must be baptized (*i.e.*, given a new name) to simplify the discourse.

Quaternion and scalar have never been used before Hamilton.

8. If I have substituted i, j, r for i, j, k, it is for two reasons: the axis r is the axis of reference of the old system; it is also the axis which bears vectors whose measures are real numbers. On the other hand k is constantly used in the expresssion $2\, k\, \pi+\theta$, and as $i\ j\ r$ often appear as exponents as well as $2\, k\pi+\theta$, there would be two k's with different meanings in the same formula.

Let then Mr. Bramg reserve his compassion for the "poor k."

9. The critic reproaches me with not having treated the bi-and tri-quaternions. When I wrote my preface, I only knew what Hamilton thought: a bi-quaternion is an expression $q_1 + q_2 \sqrt{-1}$, in which q_1, q_2 are two quaternions. Being able to define all the measures of geometrical magnitudes without appealing to anything else but the quaternions, I had reason to say that, for me, the bi-quaternions were without use.

Mr. Bramg knows the tri-quaternions no more than I did a year ago. I have the pleasure to call his attention to the following work: 'Calcul des Tri-quaternion' by G Combebiac, Paris, Gauthier-Villars, 1902: When my book was almost printed, I understood while reading the above that the *bi*-and *tri*-quaternions are not what Hamilton imagined; and that they serve to measure displacements and motions, and not geometric magnitudes. I therefore inserted on page 269 a short rectification: *I have said that they are not numbers.* Their theory belongs to Mechanics and not to Arithmetic (such as, motion and acceleration.)

10. The critic adds that the subject has not been fully expounded. I shall call his attention to the fact that I have stated in my preface, my intention to *define* numbers, and give their *principal properties.* I have not then to study them in all their *applications.* That is the object of works special for each class of numbers.

In short, Mr. Bramg has read my work very superficially: he has not seen its essential characteristic, and his review is limited to the discussion of *words* and not *ideas.*

&c. &c.

The name usually employed in England for the group of subjects treated in this book is 'Universal Algebra'. Prof. Dumont does indeed discuss "number" in his preface, but the result of the discussion can hardly be called a logical definition as a reference to the book will show. Perhaps, it would have been better if we had said that "number is not defined though an attempt is made."

Analytically, a ternion is a quaternion with a non-scalar element zero; geometrically, it is a quaternion with its axis in a given plane. There is no reason to suppose that this plane is fixed always and everywhere. $w+i\,x+j\,y$ is a ternion, as well as $w+ix+kz$ and their sum is a quaternion. The unqualified statement that the sum of two ternions is a ternion is as incorrect as the unqualified statement that the tensor, of the sum of two vectors is the sum of their tensors.

The various systems of vector analysis referred to are those developed along non-quarternion lines. We invite Prof. Dumont's attention to Heaviside's and Gibbs's work among others.

We are surprised to find that Clifford's biquaternions (invented almost half a century ago) are not known to a writer on Arithemetique Generale. A full bibliography of works on biquaternions (*alias* octonions) and on triquaternions will be found in Shaw's *Synopsis of Linear Associative Algebras*. If Prof. Dumont calls an operator like quaternion a "number," then an octonion is a number; for an octonion is related to a screw in exactly the same manner as a quaternion is related to a Hamiltonian vector.

BRAMG.

SOLUTIONS.

Questions 250, 285.

250. (N. DURAI RAJAN, B.A.) :—If in an octahedron inscribed in a sphere the three diagonals are mutually at right angles and meet in a point O, shew that

(*a*) the Nine-point circles of the eight faces lie on a sphere, which may be called the Twenty-four point sphere, and that the mid-points of the twelve edges lie on it ; (this is the analogue of the N.P. circle of a triangle, and the Twelve-point sphere of an orthocentric tetrahedron)

(*b*) the feet of the perpendiculars from O on the faces lie on a sphere (called the auxiliary sphere A) whose centre divides the line joining O and the centre of the given sphere in the ratio 1 : 2. This sphere A contains the orthocentroidal circles of the eight faces.

285. (N. DURAI RAJAN) :—Shew that the twenty-four point sphere, the auxiliary sphere, and the circum-sphere have a common radical plane, and that the point O is a point-sphere of the coaxal system.

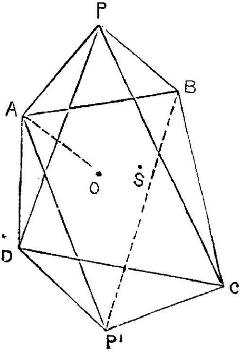
I. Solution by S. Purushottam.

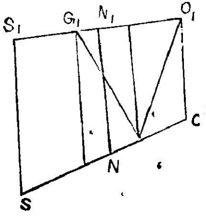
Q. 250. (*a*) Let ABCDPP′ be the octahedron inscribed in a sphere, with centre S. AC, BD and PP′ intersect in a common point O mutually at rt. ∠s.

Let S_1 O_1 be the circumcentre and orthocentre of one face, say PAB. Then it is well known that SS_1 and OO_1 are ⊥r to plane PAB and therefore to S_1 O_1.

The ⊥r at N_1, the Nine-point centre, to S_1 O_1 meets SO at its mid. pt. N.

The distances of N from the mid. pts. of the edges are equal. Therefore all the N. P. circles lie on a sphere. This sphere cuts each edge in two important points.





(*b*) Let $x = \text{OS}$ and $\text{R} =$ radius of circumsphere. Reciprocate with respect to a sphere, centre O, radius $= \sqrt{\text{OB} \cdot \text{OD}}$. The octahedron reciprocates into a rectangular parallelopiped, and the sphere into a conicoid of revolution whose one focus is at O and whose auxiliary sphere is the given sphere.

The corners of the rect. parallelopiped lie on the director sphere of the conicoid whose radius is r, say.

The inverses, with respect to the original sphere of reciprocation of the corners of the parallelopiped are the feet of the $\perp r s$ from O on the faces. Hence we must find the distance from O of the centre of the inverse, with respect to a certain sphere, of the director sphere. Let this centre be Z.

$$\text{OZ} = -\frac{(x^2-\text{R}^2)}{2}\left\{\frac{1}{r-x} \quad \frac{1}{r+x}\right\} = -\frac{x(x^2-\text{R}^2)}{r^2-x^2}$$

but $\qquad r^2 = 3\,\text{R}^2 - 2x^2$

$\therefore \qquad r^2 - x^2 = 3\,(\text{R}^2 - x^2)$

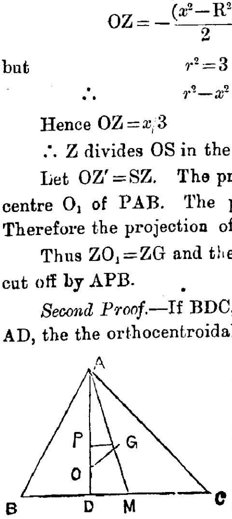
Hence $\text{OZ} = x/3$

$\therefore$ Z divides OS in the ratio of 1 : 2.

Let $\text{OZ}' = \text{SZ}$. The projection of O on a face PAB is the orthocentre O_1 of PAB. The proj. of S is the circumcentre S_1 of PAB. Therefore the projection of Z′ is the centroid:

Thus $\text{ZO}_1 = \text{ZG}$ and the mid pt. of O_1G is the centre of the circle cut off by APB.

Second Proof.—If BDC, and AD be two $\perp r$ lines and A any point in AD, the the orthocentroidal circle of ABC cuts BC in two fixed points.



Let M be the mid. pt. of BC and GP the $\perp r$ from G on AD. Let X, Y be the pts. where the ortho-centroidal ciccle cuts BC. The mid. pt. of XY is the foot of the $\perp r$ from the mid. pt. of OG, and therefore divides DM in the ratio of 1 : 2. Hence it is fixed.

Again, $\quad \text{DX.DY} = \text{DO.DP} = \frac{1}{3}\,\text{DO.DA} = \frac{1}{3}\,\text{BD.CD}$

Hence DX.DY is also fixed ThusX and Y are fixed.

If A be any where in the plane thro' D $\perp r$ to BC, the proposition still holds.

It is clear that the $\perp rs$ to faces at their ortho-centroidal centres all meet in a point which divides OS in the ratio 1 : 2.

Taking any two faces PAB, PBC, since AC is in a $\perp r$ plane to PB, the ortho. centroidal circles cut PB in the same two points. Hence it follows that a sphere passes thro' all the eight ortho-centroidal circles.

Q. 285. Let X be the pt. sphere, centre x
Y „ circumsphere, „ y
Z „ twenty-four point sphere, „ z
W „ auxiliary sphere „ w
P(m, X) „ power of a point m, with respect to X.

Then if P (m, X) : P (m, Y) $= \alpha : \beta$, the locus of m is a sphere coaxal with X and Y and dividing xy in the ratio $-\alpha : \beta$. Let m_1 be the mid. pt. of an edge of one face and O_1, G_1 its orthocentre and centroid.

$\frac{P(m_1, X)}{P(m_1, Y)} = -1$. Hence z bisects xy and Z is coaxal with X & Y.

$$\frac{P(O_1, Y)}{P(O_1, Z)} = 4, \text{ and similarly } \frac{P(G_1, Y)}{P(G_1, Z)} = 4.$$

Hence O_1G_1, lie on a sphere W coaxal with Y & Z.

The projections of x, y, z, w on a face are in a line, $O_1G_1S_1$ Hence the circle on O_1 G is a section of W by the face.

Note. The point sphere at O cuts the faces in the polar circles. Since, by Q. 250, the sections of the four spheres by any face are coaxal, the spheres have a common radical plane.

II. Solution by the Proposer.

Q. 250. Let AOA′, BOB′, COC′ be the diagonals of the octahedron. Take OA, OB, OC, as axes of coordinates.

(a) Let the equation of the circumsphere be

$$x^2+y^2+z^2+2ax+2by+2cz+d=0.$$

Let $(x_1y_1z_1)$ $(x_2y_2z_2)$ be two points which subtend an $\angle$ of 90° at O and let ξ, η, ζ be their mid-point.

$$2\xi = x_1+x_2\,;\; 2\eta = y_1+y_2\,;\; 2\zeta = z_1+z_2$$

and
$$\Sigma x_1^2 + \ldots \ldots 2ax_1 + \ldots \ldots +d=0$$
$$\Sigma x_2^2 + \ldots \ldots 2ax_2 + \ldots +d=0.$$

Adding and substituting

$$4(\xi^2+\eta^2+\zeta^2)-2x_1x_2+4(a\xi+b\eta+c\zeta)+2d=0$$

i.e. $$2(x^2+y^2+z^2)+2ax+2by+2cz+d=0$$

which is a sphere.

The mid. pts. of AB, BC......lie on this sphere.

Therefore the Nine point circles lie on this sphere.

(*b*). If OP, OQ, OR be three mutually $\perp r$ lines meeting the sphere in PQR, find the locus of the $\perp r$ from O on PQR.

Let $lx+my+nz=p$, be the equation of the plane PQR.

The cone

$$(x^2+y^2+z^2)+(2ax+2by+2cz)\frac{lx+my+nz}{p}+d\left(\frac{lx+my+nz}{p}\right)^2=0$$

contains an infinite number of three $\perp_r$ generators.

$$\therefore \quad A+B+C=0$$

$$\therefore \quad 3p^2+2p(al+bm+cn)+d=0$$

If the co-ordinates of the foot of $\perp_r$ be (x,y,z)

$$p^2=x^2+y^2+z^2 \text{ and } x=pl,\ y=pm,\ z=pn.$$

$$\therefore \quad 3p^2+2p(ax+by+cz)+d=0.$$

$$\therefore \quad 3(x^2+y^2+z^2)+2ax+2by+2cz+d=0$$

The centre, H, of this sphere divides OS in the ratio 1 : 2, and it passes thro' O_1 the orthocentre of a face. The foot of $\perp_r$ from it on a face is the mid. pt. of O_1G_1

Therefore the auxiliary sphere passes thro' G_1 also, and therefore 'A' contains the eight ortho-centroidal circles.

N.B.—It can also be proved that 'A' is the auxiliary sphere of a prolate spheroid, one focus at O; and the distance between the centroids of two opposite faces is constant.

Q. 285. The equation of the point-sphere is $x^2+y^2+z^2=0$.

Hence the four spheres are coaxal.

Note.—The Tetrahedral analogue is given in Wolstenholme's Problems.

Question 270.

(S. NARAYANAN B.A., L.T.) :—Shew that the locus of the orthopole of a straight line passing through the incentre of a triangle ABC is a conic inscribed in the excentral triangle of the medial triangle of ABC having its centre at the circumcentre of that triangle.

Remarks by S. Purushottam.

Let P be a point on the circumcircle of ABC. Then the Simson line of P cuts the orthopolar conic of any point I in the orthopoles of PI, and QIR, where

$$arc\ PQ+arc\ PR=arc\ PA+arc\ PB+arc\ PC.$$

When I is the in centre of ABC, and P the middle point of the arc BC, PI concides with QIR, and the pedal line of P *touches* the conic at the orthopole of PI or PA.

Now, this orthpole is readily seen to be the intersection of the perpendiculars from A and the mid-point of BC on BC and PA respectively. Well known geometrical considerations lead to the other properties of the conic stated in the question.

Question 275.

(M. BHIMASENA RAO) :—Show that the foci of the three parabolas each touching two of the three sides of a triangle, and also the join of their mid-points at the point where this join is intersected by the corresponding side of the orthocentric triangle, are collinear.

Solution by S. Narayanan, B.A., L.T., and N. Sankara Aiyar.

Let E and F be the midpoints of AC, AB of the triangle and let K be the intersection of EF with the corresponding side of the orthocentris triangle. Then if $\sqrt{l\alpha}+\sqrt{m\beta}+\sqrt{n\gamma}=0$, be any inconic of the triangle AEF, the condition for a parabola gives

$$\frac{l}{a}+\frac{m}{b}+\frac{n}{c}=0 \quad \ldots\ldots\ldots\ldots \quad (1)$$

Also, since the coordinates of K, when referred to the triangle AEF are easily seen to be (0, sin $\overline{C-A}$, sin $\overline{A-B}$), the condition that the parabola should pass through K gives

$$m \sin \overline{C-A} = n \sin \overline{A-B} \quad \ldots\ldots \quad (2).$$

Hence, combining (1) & (2) l, m, n are proportional to

$$a^2 \sin \overline{B-C}\ ,\ bc \sin \overline{A-B}\ ,\ bc \sin C-A.$$

The focus of the inscribed parabola being $\left(\frac{a^2}{l}, \frac{b^2}{m}, \frac{c^2}{n}\right)$ we get for the coordinates of the focus, when referred to the triangle AEF

$$\frac{a}{b^2-c^2}, \frac{b}{a^2-b^2}, \frac{c}{c^2-a^2}.$$

Therfore the coordinates of the focus when referred to the triangle ABC, are

$$a(2a^4+b^4+c^4-2a^2b^2-2a^2c^2),\ b(c^2-a^2)(c^2-b^2),\ c(b^2-a^2)(b^2-c^2).$$

Hence writing down from symmetry the coordinates of the foci of the other two parabolas, we see that the three foci lie on the line

$$\alpha \sin \overline{B-C}+\beta \sin \overline{C-A}+\gamma \sin \overline{A-B}=0,$$

which is the join of the circumcentre and the symmedian point.

Question 301 (*a*).

(A. C. L. Wilkinson, M.A, F.R.A.S.) ;—Prove that the following expressions are true, the inverse tangents in all cases, denoting angles between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$:

$$(1) \sum_1^{o} \tan^{-1} \frac{4\sqrt{2}(n^2+3)}{(n^2-1)^2} = 2\pi - \tan^{-1}\frac{2\sqrt{2}}{3}$$

$$(2) \sum_1^{\infty} \tan^{-1} \frac{8\sqrt{2}(n^2-6)}{(n^2+2)^2} = \tan^{-1} 12\sqrt{2} + \tan^{-1}\frac{6\sqrt{2}}{3}$$

$$(3) \sum_1^{\infty} \tan^{-1} \frac{4n^2}{n^2-4} = \frac{\pi}{2}$$

$$(4) \sum_1^{\infty} \tan^{-1} \frac{16(n^2-8)}{n^2(n^2+16)} = \frac{\pi}{4}$$

$$(5) \sum_1^{\infty} \tan^{-1} \frac{4(n^2-A^2)}{(n^2+A^2)^2-4} = \tan^{-1}\frac{2}{A^2}$$

Remarks by the Proposer.

I may note that originally, I proceeded to obtain some of the results given by a method identical with that given in the solution by Mr. R. Srinivasan (*vide*: J. I. M.S, Vol. III, p. 244.) It is however clear that this method does not show that the inverse tangents are definitely determined as represented in the question.

The following processes are free from all ambiguity.

Consider the identity

$$\tan^{-1}\frac{a(n-1)}{n^2-2n+b} - \tan^{-1}\frac{a(n+1)}{n^2+2n+b} = \tan^{-1}\frac{2a(n^2-b)}{n^4+n^2(a^2+2b-4)+b^2-a^2}$$

(1) Write $a=2\sqrt{2}$, $b=-3$, and this becomes

$$\tan^{-1}\frac{2\sqrt{2}(n-1)}{n^2-2n-3} - \tan^{-1}\frac{2\sqrt{2}(n+1)}{n^2+2n-3} = \tan^{-1}\frac{4\sqrt{2}(n^2+3)}{(n^2-1)^2}$$

or $v_n - v_{n+2} = u_n$, say .

This holds good for the inverse tangents having positive acute values, provided $n \geq 3$.

Now, $u_1 = \frac{\pi}{2}$, $u_2 = \tan^{-1}\frac{28\sqrt{2}}{9}$, $u_3 = \frac{\pi}{2} - v_5$, $u_4 = v_4 - v_6$, etc.

whence on addition $\sum_1^{\infty} u_n = \pi + \tan^{-1}\frac{28\sqrt{2}}{9} + v_4$

$$= \pi + \tan^{-1}\frac{28\sqrt{2}}{9} + \tan^{-1}\frac{6\sqrt{2}}{5}$$

$$= 2\pi - \tan^{-1}\frac{2\sqrt{2}}{5},$$

which is result (1).

[The value of u_1 may equally well be $-\frac{\pi}{2}$, according to the conditions of the question. Ed.]

(3) Again, write $a=$ ', $b=0$, we have

$$\tan^{-1}\frac{4n^2}{n^4-4}=\tan^{-1}\frac{2(n-1)}{n^2-2n}-\tan^{-1}\frac{2(n+1)}{n^2+2n}$$

or

$$u_n \;=\; v_n-v_{n+2}$$

This holds good provided $n\geq 2$.

Here $u_1=-\tan^{-1}\frac{4}{3}$, $u_2=\frac{\pi}{2}-v_4$, $u_3 \;=v_3-v_5$ etc.

whence $$\sum_1^\infty u_n=\frac{\pi}{2}-\tan^{-1}\frac{4}{3}+v_3=\frac{\pi}{2}\text{;}$$ which is result (3).

Again, consider the identity

$$\tan^{-1}\frac{a(n-2)}{n^2-4n+b}-\tan^{-1}\frac{a(n+2)}{n^2+4n+b}=\tan^{-1}\frac{4a(n^2-b)}{n^4n^2(a^2+2b-16)+b^2-4a^2}.$$

(2) Write $a=2\sqrt{2}$, $b=6$, we obtain

$$\tan^{-1}\frac{2\sqrt{2}(n-2)}{n^2-4n+6}-\tan^{-1}\frac{2\sqrt{2}(n+2)}{n^2+4n+6}=\tan^{-1}\frac{8\sqrt{2}(n^2-6)}{(n^2+2)^2}$$

or $v_n-v_{n+4}=u_n$, holding good provided $n\geq 2$ for the acute and positive values of the angles.

Hence $u_2=-v_6$, $u_3=v_3-v_7$, etc., gives on summation

$$\sum_2^\infty=v_3+v_4+v_5=2\tan^{-1}\frac{2\sqrt{2}}{3}+\tan^{-1}\frac{6\sqrt{2}}{11}$$

$$=\tan^{-1}12\sqrt{2}+\tan^{-1}\frac{6\sqrt{2}}{11},$$ which is result (2)

[N.B.—The limits are 2 to ∞, and not 1 t ∞, as printed.]

(4) Again write $a=4$, $b=8$, we obtain

$$\tan^{-1}\frac{4(n-2)}{n^2-4n+8}-\tan^{-1}\frac{4(n+2)}{n^2+4n+8}=\tan^{-1}\frac{16(n^2-8)}{n^2(n^2+16)},$$

or $v_n-v_{n+4}=u_n$, holding good for $n\geq 2$.

Now $u_1=-\tan^{-1}\frac{4}{5}-\tan^{-1}\frac{12}{13}$, $u_2=-v_6$, $u_3=\tan^{-1}\frac{4}{5}-v_7$, $u_4=\frac{\pi}{4}-v_8$,

$u_5=\tan^{-1}\frac{12}{13}-v_7$, etc.,

whence on summation $\sum_1^\infty u_n=\frac{\pi}{4}$, which is result (4).

(5) Lastly, in the first identity write $a=2$, $b=A^2$.

Then $\tan^{-1}\dfrac{4(n^2-A^2)}{(n^2+A^2)^2-4}=\tan^{-1}\dfrac{2(n-1)}{n^2-2n+A^2}-\tan^{-1}\dfrac{2(n+1)}{n^2+2n+A^2}$.

or $u_n=v_n-v_{n+2}$, holding good for all values of n when A^2 is positive.

Hence $u_1=\tan^{-1}\dfrac{4(1-A^2)}{A^4+2A^2-3}=\tan^{-1}\left(-\dfrac{4}{A^2+3}\right)$

$u_2=\tan^{-1}\dfrac{2}{A^2}-v_4$, $u_3=\tan^{-1}\dfrac{4}{A^2+3}-v_5$, $u_4=v_4-v^{40}$ etc ,

On summation $\sum_1^\infty u_n=\tan^{-1}\dfrac{2}{A^2}$, which is result (5).

Question 307.

(M. T. Naraniengar, M.A.) :—Invert a given triangle so that the inverse triangle may be *congruent* to the original triangle.

Solution by N. Durairajan.

The inverse of a triangle ABC with respect to a point O is easily seen to be similar to the pedal triangle of O with respect to ABC. Hence, we have to find the positions of O such that the pedal triangle may be similar to ABC.

Now, since the Brocard angle of the pedal triangle of O is equal to the Brocard angle of ABC, O must lie on the Brocard Circle of ABC. The six positions of O are got by comparing the pedal triangle successively with ABC, BCA, CAB ; ACB, BAC, CBA. The points are :—the circumcentre, the two Brocard points, and the intersections of AK, BK, CK with the Brocard Circle, K being the symmedian point.

The radius ρ of inversion, for congruence, is given by the equation :

$\rho^2=a'yz/a=b'zx/b=c'xy/c$, where $x=OA$, $y=CB$, $z=OC$; and a', b', c' stand for a, b, c in different orders, according to the position of O. [*Vide* : J. I. M. S., Vol. IV, No 1., p. 23 § 10.]

Question 316.

(V. Ramaswami Aiyar, M. A.) :—O is the centre and PQ any diameter of the circle inscribed in an equilateral triangle ABC. D is a point

in BC such that AO and OD are equally inclined to PQ. Prove that the feet of the perpendiculars from O on the sides of the quardrilateral APDQ lie on a circle which touches the inscribed circle.

Solution and remarks by M. Bhimasena Rao.

Consider a circle X touching the circumcircle of a triangle ABC and cutting BC in D and D′ Let E, E′ be points on X such that ED and E′ D′ touch the circumcircle at P and P′. Taking ABC as the triangle of reference, if P and P′ be $\left(\frac{1}{l},\frac{1}{m},\frac{1}{n}\right)$ and $\left(\frac{1}{l'},\frac{1}{m'},\frac{1}{n'}\right)$ the circle X is

$$(al^2\alpha+bm^2\beta+cn^2\gamma)(al'^2\alpha+bm'^2\beta+cn'^2\gamma)+\lambda\alpha(p\alpha+q\beta+r\gamma)=0,$$

for some value of λ, where EE′ is $p\alpha+q\beta+\nu\gamma=0$.

If EE′ passes through A, $p=0$ and the radical axis of X and the circum circle is

$$al^2l'^2\alpha+bm^2m'^2\beta+cn^2n'^2\gamma=0.$$

Since the radical axis touches the circumcircle, we have

$$all'\pm bmm'\pm cnn'=0. \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Also P P′ is $all'\alpha+bmm'\beta+cnn'\gamma=0$, and it is therefore evident that PP′ passes through the incentre or an excentre.

Taking the incircle, or the corresponding excircle, as the circle of reciprocation, we get the following more general theorem :—

O is the circumcentre of a triangle ABC and PQ is any diameter of the inconic whose centre is the nine-point centre. If D is a point in BC such that the angles AOP and DOQ are equal or supplementary, the feet of the perpendiculars from O on the sides of the quadrilateral APDQ lie on a circle which touches the nine-point circle.

Additional Solution by N. B. Pendse.

Question 319.

(J. C. Swaminarayan, M.A.) :—Determine a point X on a horizontal axis OX from which a particle can be projected with the least velocity so as to pass through two given points (ξ, η) and (ξ', η') in the same vertical plane as OX and shew that

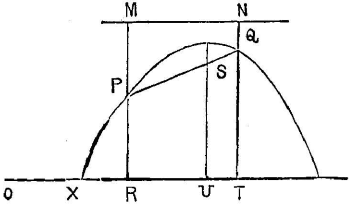
$$OX=\frac{1}{2\rho}\left[(\xi+\xi')\rho+(\eta-\eta')(\xi-\xi')-(\xi-\xi')\sqrt{[(\rho+\eta+\eta')^2-4\eta\eta']}\right]$$

where $\rho=\{(\xi-\xi')^2+(\eta-\eta')^2\}^{\frac{1}{2}}$

Solution by V. B. Naik, N. Sankara Aiyar and the Proposer.

Let P and Q be the points (ξ, η) and (ξ', η').

The least velocity of projection from a point on the X-axis is that which makes the height of the directrix of the trajectory through P and Q least; *i.e.*, the focus of the parabola must be on PQ.



Let h be the height of the directrix above the x—axis, and let S be the focus.

Then $SP = PM = h - \eta$, $SQ = QN = h - \eta'$

$$\therefore\ \rho = SP + SQ = 2h - \eta - \eta' \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{I}$$

$$\text{Also, } OU = \frac{\xi(h-\eta'') + \xi'(h-\eta)}{h-\eta+h-\eta'} = \frac{h(\xi'+\xi-\xi\eta'-\xi'\eta}{\rho} \ldots \quad \ldots \quad \text{II}$$

$$\text{and} \qquad SU = \frac{\eta(h-\eta') + \eta'(h-\eta)}{h-\eta+h-\eta'} = \frac{h(\eta+\eta')-2\eta\eta'}{\rho} \quad \ldots \quad \ldots \quad \text{III}$$

Now $(OU - OX)^2 + SU^2 = SX^2 = h^2$

$$\therefore \qquad OX = OU - \sqrt{(h^2 - SU^2)} \quad \ldots \quad \ldots \quad \ldots \quad \text{IV}$$

Also $h^2 - SU^2 = (h + SU)(h - SU)$, which by using I and III,

$$= \left[\frac{h(\rho+\eta+\eta')-2\eta\eta'}{\rho}\right]\left[\frac{h(2h-\eta-\eta')-h(\eta+\eta')+2\eta\eta'}{\rho}\right]$$

$$= \frac{1}{\rho^3}(4h^2 - 4\eta\eta')(h-\eta)(h-\eta')$$

$$= \frac{1}{4\rho^2}(4h^2 - 4\eta\eta')(\xi'-\xi)^2 \quad \ldots \quad \ldots \quad \ldots \quad \text{V}$$

$$\text{Again } OU = \frac{h(\xi+\xi')-\xi\eta'-\xi'\eta}{\rho} = \frac{(\rho+\eta+\eta')(\xi+\xi')-2\xi\eta'-2\xi'\eta}{2\rho}$$

$$= \frac{\rho(\xi+\xi')+(\xi-\xi')(\eta-\eta')}{2\rho} \quad \ldots \quad \ldots \quad \text{VI}$$

Substituting V and VI in IV, the result follows.

QUESTIONS FOR SOLUTION.

Addendum.—The last two questions on page 40 (J.I.M.S., Vol. IV) should be numbered 354 and 355 respectively.

356. (A. C. L. Wilkinson, M.A., F.R.A.S.) :—If a tetrahedron ABCD is such that

$$\cos BAD \cos BCD = \cos ABC \cos ADC,$$

the shortest distances between the pairs of opposite edges (AB, CD) (BC, AD) intersect.

357. (A. C. L. Wilkinson, M.A., F.R.A.S.,) :—If the middle points of the shortest distances between opposite edges of a tetrahedron and its centroid are coplanar, shew that two of the shortest distances must intersect.

358. (S. Ramanujan) :—If n is a multiple of 4, excluding zero, shew that

$$1^{n-1} \operatorname{sech}\frac{\pi}{2} - 3^{n-1} \operatorname{sech}\frac{3\pi}{2} + 5^{n-1} \operatorname{sech}\frac{5\pi}{2} - + \ldots = 0.$$

359. (S. Ramanujan) :—If $\sin(x+y) = 2 \sin\frac{x-y}{2}$, and $\sin(y+z) = 2\sin\frac{y-z}{2}$, prove that

$$\sqrt[4]{\tfrac{1}{2}} \sin x \cos z + \sqrt[4]{\tfrac{1}{2}} \cos x \sin z = \sqrt[12]{\sin 2y},$$

and verify the result when

$$\sin 2x = (\sqrt{5}-2)^{2}(4+\sqrt{15})^{2};\ \sin 2y = (\sqrt{5}-2);\ \text{and}$$
$$\sin 2z = (\sqrt{5}-2)^{2}(4-\sqrt{15})^{2}.$$

360. (J. C. Swaminarayan, M.A.) :—G is the centroid of a triangle ABC, and G′ that of the triangle formed by the images of any point P on the circumcircle in the sides of ABC. Shew that GG′ is perpendicular to the Simson's line of P.

361. (J. C. Swaminarayan, M.A.) :—Prove that

$$a(b-c)(b+c-a)(b^2+c^2-a^2) + \text{two similar expressions} = 0.$$

362. (S. Narayan, B.A., L.T.) :—ABC is a triangle whose symmedian point is K. PQR is the reciprocal polar of ABC with respect to K, the modulus being the power of K with respect to the circumcircle. Show that the polar conic of K with respect to PQR is the locus of points whose pedal triangles with respect to ABC, have their centroids on the circumcircle of ABC.

363. (SELECTED) :—Give a proof of the correctness of the Russian peasant method of multiplication, which may be described as follows :—

To multiply a by b, write down $a \times b$. Under a write the exact or lower quotient obtained by dividing it by 2, under this quotient write the similar quotient obtained by dividing it by 2, and so on until the quotient 1 is obtained. Under b write its double, under this its double, and so on until there are as many numbers in the second column as in the first. Next, add the numbers in the second column which correspond to *odd* numbers in the first column. The result is the product of a and b.

364. (ZERO) :—Explain a simple method of rationalizing the equation

$$x^{\frac{4}{5}} + a\, x^{\frac{3}{5}} + b\, x^{\frac{2}{5}} + c\, x^{\frac{1}{5}} + d = 0,$$

and discuss the rationalized form of the general equation

$$f\left(x^{\frac{1}{r}}\right) = 0.$$

365. (T. SUNDARA RAO) :—The Jerabek points J_1 J_2 of a triangle ABC are such that their distances from the sides of the triangle measured parallel to (AB, BC, CA) and (BA, AC, CB), respectively, are all equal (*vide* : J.I.M.S., Vol. IV, page 18). Shew that associated with these points is a third point J_3 such that if parallels are drawn from it to the sides of the triangle, the parallels drawn *in one sense* are each equal to the parallels drawn to the same side *in the opposite sense* from J_1 or J_2.

366. (M. KANNAN, B.A., L.T.) :—In a triangle ABC, PP′ is a diameter of the circumcircle, and the Simson's lines of P and P′ intersect at ω. Show that the length of the perpendicular from ω on PP′ is $2\,R \cos \alpha \cos \beta \cos \gamma$, where α, β, γ are the inclinations of PP′ to the sides of the triangle.

367. (M. BHIMASENA RAO) :—Show that the normal co-ordinates of the double point of ABC, and triangle whose angular points are (l_1, m_1, n_1); (l_2, m_2, n_2); (l_3, m_3, n_3), are

$$(al_1 + bm_1 + cn_1)\ (al_2 l_3 + bl_2 m_3 + cl_3 n_2), \text{ etc };$$

and hence or otherwise that the double points of two triangles in perspective is the centre of the conic with respect to which the triangles are conjugate.

368. (M. BHIMASENA RAO) :—Through the median point of a triangle a line is drawn cutting the circumcircle in P and Q. Show that the polars of P and Q with respect to the triangle are respectively perpendicular to the pedal lines of Q and P.

369. (A. C. L. Wilkinson, M.A., F.R.A.S.) :—If a surface be inverte l from any point O, and if ρ_1, ρ_2 ; ρ_1', ρ_2' be the principal radii of curvature at any two corresponding points P, P′, shew that

$$OP' \left(\frac{1}{\rho_1'}-\frac{1}{\rho_2'}\right) = OP \left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)$$

370. (K. J. Sanjana, M. A.) :—Solve graphically the equation $a \tan \theta + b \sin \theta = c$, which a, b, c are numerical constants.

371. (K. J. Sanjana, M. A.) :—I, I_1, I_2, I_3, are the centres of the four contact circles of a triangle, and N, N_1, N_2, N_3 the nine-point centres of the triangles formed by their respective points of contact : prove that NI, N_1I_1, N_2I_2, N_3I_3 are concurrent.

372. (D. D. Kapada, M.A., B.Sc.) :—A rectangular hyperbola has contact of the highest order possible with a given curve. Prove that the semi-transverse axis of the hyperbola is

$$\rho \left\{1+\frac{1}{9}\left(\frac{d\rho}{ds}\right)^2\right\}^{-\frac{3}{4}}.$$

373. (D. D. Kapadia, M.A., B.Sc.) :—Prove that

$$\begin{vmatrix} x-a_1, & a_1, & a_1, & \cdots & a_1 \\ a_2, & x-a_2, & a_2, & \cdots & a_2 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_n, & a_n, & & \cdots & x-a_n \end{vmatrix} \equiv f(x). \left\{1+\sum\left(\frac{a_r}{x-2a_r}\right)\right\}$$

where $f(x)$ stands for $(x-2a_1)(x-2a_2)\ldots(x-2a_n)$.